Secular series and renormalization group for amplitude equations

Eleftherios Kirkinis*
Applied Mathematics, University of Washington, Seattle, Washington 98195-2420, USA
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We have developed a technique that circumvents the process of elimination of secular terms [L.-Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **54**, 376 (1996)] and reproduces the uniformly valid approximations, amplitude equations, and first integrals. The technique is based on a rearrangement of secular terms and their grouping into the secular series that multiplies the constants of the asymptotic expansion. We illustrate the technique by deriving amplitude equations for standard nonlinear oscillator and boundary-layer problems.

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I. INTRODUCTION

Inspired by the statistical mechanics of critical phenomena and quantum field theory [1], Chen, Goldenfeld, and Oono (CGO) developed a new method, the renormalization group (RG) for the asymptotic solution of differential equations [2], henceforth referred to as the CGO renormalization method. In statistical mechanics the renormalization group extracts structurally stable features of systems which are insensitive to details. In the asymptotic solution of differential equations the RG eliminates nonuniformities by a resummation process. The asymptotic solutions derived in [2] were in many cases superior to those obtained by standard multiple-scale, Wentzel-Kramers-Brillouin (WKB), or matched asymptotic expansions. In some cases the complete closed form solutions can be captured.

Despite the merit of the CGO RG and related methods [2–4], they rely on a process of elimination and the derivation of amplitude equations that can be unwieldy at times. Our technique provides the same asymptotic expansions and amplitude equations as the standard RG approach [2,4]. Its merit lies on using a near identity transformation based on the secular series, thus effectively circumventing the painstaking elimination process, of determining the eliminative sequence $\{Z_i\}_{i=1}^{\infty}$. It exposes the underlying structure of the RG, avoids the introduction of additional structures such as secondary parameters [2] or the use of envelopes of families of curves [3], and at times directly provides an asymptotic solution through a first integral of the amplitude equations. The critical step is to recognize that the naive asymptotic expansion can be rearranged so that the secular terms are grouped into the secular series which in turn multiplies each constant appearing in the asymptotic expansion. The naive perturbation expansion is defined in this paper as one having the form $y=y_0(A,t)+\epsilon y_1(A,t)+\epsilon^2 y_2(A,t)+\cdots$ where A denotes a set of integration constants arising in the zeroth-order solution and t a generic independent variable.

We emphasize that the technique introduced in this Brief Report gives results identical to those of the CGO RG. This is not surprising since it is well known from field theory that other methods exist that provide results identical to those of the RG.

*kirkinis@amath.washington.edu

In Sec. II we briefly describe the technique and connect it with the literature on the renormalization group. Section III illustrates the methodology with three nonlinear examples.

II. GENERAL INTRODUCTION TO OUR TECHNIQUE

Applying a naive perturbation expansion to the solution of a differential equation might lead to nonuniformities, for example, when $t \geqslant O(1/\epsilon)$ resulting from a secular term in the perturbation solution of the form ϵt. The CGO RG eliminates these secular terms by replacing the constant of integration A in the zeroth-order solution by a slowly varying amplitude $\mathcal{A}(t)$ by resorting to the near-identity transformation

$$A = \mathcal{A}[1 + \epsilon Z_1(\mathcal{A},t) + \epsilon^2 Z_2(\mathcal{A},t) + O(\epsilon^3)], \tag{1}$$

thus requiring the determination of the unknown terms Z_i, $i=1,2,\ldots$.

Instead, let us define the secular series as

$$y_p = [1 + \epsilon y_{1p}(A,t) + \epsilon^2 y_{2p}(A,t) + O(\epsilon^3)], \tag{2}$$

where $\{y_{ip}\}_{i=1}^{\infty}$ is the sequence of secular polynomial factors that arise in the successive particular solutions and are responsible for the secular behavior of the perturbation expansion, and relate A and $\mathcal{A}(t,\epsilon)$ through the near identity transformation

$$\mathcal{A}(t,\epsilon) = A[1 + \epsilon y_{1p} + \epsilon^2 y_{2p} + O(\epsilon^3)], \tag{3}$$

i.e., let $\mathcal{A}(t,\epsilon) \equiv A y_p$ or $A \equiv \mathcal{A}(t,\epsilon) y_p^{-1}$ (the secular series will be enclosed in square brackets throughout this Brief Report). The motivation behind the introduction of (3) to replace the constants, will become evident in the following sections where we investigate the asymptotic solutions to specific nonlinear problems.

To obtain the amplitude equation we follow [2] and differentiate $A=\mathcal{A}y_p^{-1}$ with respect to time t. Since A is a constant this step leads to a first-order ordinary differential equation for the slowly varying amplitude, $\mathcal{A}$

$$\frac{d\mathcal{A}}{dt} = \mathcal{A} y_p^{-1} \frac{dy_p}{dt}. \tag{4}$$

Substituting the secular series expansion (2) for y_p in the above expression and rearranging leads to the following form of the amplitude equation:

$$\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{dt} = \epsilon\frac{dy_{1p}}{dt} + \epsilon^2\left(\frac{dy_{2p}}{dt} - y_{1p}\frac{dy_{1p}}{dt}\right) + \epsilon^3\left(\frac{dy_{3p}}{dt} - y_{1p}\frac{dy_{2p}}{dt} - y_{2p}\frac{dy_{1p}}{dt} + y_{1p}^2\frac{dy_{1p}}{dt}\right) + O(\epsilon^4). \quad (5)$$

The above equation is central to most RG calculations and forms the penultimate step prior to establishing the RG asymptotic expansion. In this paper we also show that by direct integration with respect to the time variable one obtains a first integral of the amplitude Eq. (5) in the form

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + \epsilon y_{1p} + \epsilon^2\left(y_{2p} - \frac{y_{1p}^2}{2}\right) + \epsilon^3\left(y_{3p} - y_{1p}y_{2p} + \frac{y_{1p}^3}{3}\right) + O(\epsilon^4). \quad (6)$$

At this point we must emphasize another major difference with the CGO RG: Here the quantities in (5) and (6) depend on the constant A. Thus, it is necessary to substitute $A = \mathcal{A}y_p^{-1}$ in the above expressions to obtain the explicit dependence on the amplitude $\mathcal{A}$. This final step is not required by the CGO RG.

The solution technique introduced in this Brief Report commences with (6). In certain cases it gives the asymptotic solution directly. This is the case for a linear singularly perturbed differential equation but also in those cases where the nonlinearity plays the role of a restoring force. In the latter case, reverting to suitable curvilinear coordinates (see the third example of Sec. III) the first integral directly provides the asymptotic solution [relations (21)–(24) of this Brief Report and Refs. [5–7]]. In general one needs to resort to the amplitude equation (5) and this is reflected in the solution process employed in the first two examples of Sec. III. In either case the only information needed in order to construct the amplitude Eq. (5) or its first integral (6) are the particular solutions of the hierarchy of equations.

III. ILLUSTRATIVE NONLINEAR EXAMPLES

A. Nonlinear boundary-layer problem

Consider the nonlinear equation

$$\epsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y^2 = 0, \quad y(0)=0, \quad y(1)=0, \quad \epsilon \to 0+. \quad (7)$$

As there is a boundary layer at the origin we introduce stretched coordinates $X=x/\epsilon$, $y(x)=Y(X)$, substitute into (7) and expand the resulting equation $Y''+2Y'=-\epsilon Y^2$ in a naive perturbation series $Y=Y_0+\epsilon Y_1+\epsilon^2 Y_2+\cdots$ which leads to the hierarchy of linear equations

$$Y_0''+2Y_0'=0, \quad Y_1''+2Y_1'=-Y_0^2, \quad Y_2''+2Y_2'=-2Y_0Y_1, \ \ldots ,$$

and the corresponding particular solutions Y_k for $k\geq 1$,

$$Y_0 = A + Be^{-2X},$$

$$Y_1 = -\tfrac{1}{2}A^2X + ABXe^{-2X} - \tfrac{1}{8}B^2e^{-4X},$$

$$Y_2 = -\tfrac{1}{4}A^3X + \tfrac{1}{4}A^3X^2 + \tfrac{1}{4}A^2B(X^2+X)e^{-2X} + \left(-\tfrac{5}{32}AB^2 - \tfrac{1}{4}B^2AX\right)e^{-4X} + \tfrac{1}{96}B^3e^{-6X}. \quad (8)$$

The main idea conveyed in this Brief Report results from the observation that the naive perturbation expansion can be re-expressed in the form

$$\begin{aligned} Y &= A\left[1 - \epsilon\frac{1}{2}AX + \epsilon^2\left(-\frac{1}{4}A^2X + \frac{1}{4}A^2X^2\right) + O(\epsilon^3)\right] \\ &+ B\left[1 + \epsilon AX + \epsilon^2\frac{A^2}{4}(X^2+X) + O(\epsilon^3)\right]e^{-2X} \\ &- \epsilon\frac{1}{8}B^2[1+\epsilon AX + O(\epsilon^2)]^2e^{-4X} \\ &+ \epsilon^2\left(-\frac{5}{32}A\left[1 - \epsilon\frac{1}{2}AX + O(\epsilon^2)\right] \times B^2[1+\epsilon AX+O(\epsilon^2)]^2e^{-4X} + \frac{1}{96}B^3[1+\epsilon AX+O(\epsilon^2)]^3e^{-6X}\right) + O(\epsilon^4), \end{aligned} \quad (9)$$

where the expressions in the square brackets are the secular series y_{Ap} and y_{Bp}; apparently, all the incoherent secular terms in (8) have been rearranged and grouped into the orderly expressions in the square brackets that appear in (9). The reader may verify that upon expanding the square brackets in (9) one can recover all the secular terms of (8). Thus, defining the slowly varying amplitudes

$$\mathcal{A} = A\left[1 - \epsilon\frac{1}{2}AX + \epsilon^2\left(-\frac{1}{4}A^2X + \frac{1}{4}A^2X^2\right) + O(\epsilon^3)\right], \quad (10)$$

$$\mathcal{B} = B\left[1 + \epsilon AX + \epsilon^2\frac{A^2}{4}(X^2+X) + O(\epsilon^3)\right], \quad (11)$$

substituting into (9) and reverting to the original variable x leads to the following asymptotic expansion:

$$y(x;\epsilon) = \mathcal{A} + \mathcal{B}e^{-2x/\epsilon} - \epsilon\frac{1}{8}(\mathcal{B}e^{-2(x/\epsilon)})^2 + \epsilon^2\left(-\frac{5}{32}\mathcal{A}(\mathcal{B}e^{-2(x/\epsilon)})^2 + \frac{1}{96}(\mathcal{B}e^{-2(x/\epsilon)})^3\right) + O(\epsilon^3). \quad (12)$$

It remains to express the amplitudes $\mathcal{A}$ and $\mathcal{B}$ elegantly, rather than in their awkward form (10) and (11): Calculate the cumulants of the sequence $\{y_{ip}\}_{i=1}^{\infty}$ as in Sec. II and substitute into (6) to obtain the algebraic relations

$$\ln \mathcal{A}(X) = \ln \mathcal{A}(0) - \epsilon\frac{1}{2}AX + \epsilon^2\frac{A^2}{8}(X-2X^2) + O(\epsilon^3),$$

$$\ln \mathcal{B}(X) = \ln \mathcal{B}(0) + \epsilon AX + \epsilon^2 \frac{1}{4} A^2 (X - X^2) + O(\epsilon^3).$$

Differentiating with respect to the stretched variable X, and using the identity $A = \mathcal{A}[1+\epsilon\frac{1}{2}AX + O(\epsilon^2)]$ [i.e., inverting (10)] leads to the pair of (weakly) coupled differential equations

$$\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{dX} = -\epsilon\frac{1}{2}\mathcal{A} - \epsilon^2\frac{1}{4}\mathcal{A}^2 + O(\epsilon^3), \tag{13}$$

$$\frac{1}{\mathcal{B}}\frac{d\mathcal{B}}{dX} = \epsilon\mathcal{A} + \epsilon^2\frac{1}{4}\mathcal{A}^2 + O(\epsilon^3), \tag{14}$$

which determine the amplitudes that appear in expansion (12). The solution of (13) plays the role of the outer expansion, most familiar from the method of matched asymptotic expansions. $\mathcal{A}$ in (13) can be obtained termwise as a function of X by a regular perturbation technique, while $\mathcal{B}$ in (14) follows by solving a linear equation. Note that the above equations are identical to those derived with the CGO renormalization approach [6] employing the near-identity transformation (1).

At this point we would like to emphasize that even though the observation that led to the formation of Eq. (9) requires a certain degree of familiarity with the problem at hand, once the significance of this step is realized by the reader, it does not need to be performed for the solution of other problems. Thus, one only needs to identify the secular terms that multiply the fundamental set of solutions of the zeroth-order equation (here these were the functions 1 and e^{-2X}), form the secular series and then collect all the nonsecular terms into an expression of the form (12). The remaining secular terms appearing in the hierarchy (8) (i.e., those multiplying e^{-4X}, e^{-6X}, etc.) need not be considered further as they are just a by-product of expanding the expressions appearing in (9).

B. Van der Pol oscillator

We consider the problem introduced by Van der Pol,

$$\ddot{y} + y = \epsilon\dot{y}(1-y^2), \quad y(0)=1, \quad \dot{y}(0)=0, \quad \epsilon \to 0+. \tag{15}$$

We expand the solution in a power series of $\epsilon y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$ that leads to a hierarchy of equations

$$\ddot{y}_0 + y_0 = 0, \quad \ddot{y}_1 + y_1 = \dot{y}_0(1-y_0^2),$$

$$\ddot{y}_2 + y_2 = \dot{y}_1(1-y_0^2) - 2y_0y_1\dot{y}_0,$$

$$\ddot{y}_3 + y_3 = \dot{y}_2(1-y_0^2) - 2y_0y_1\dot{y}_1 - \dot{y}_0(2y_0y_2 + y_1^2),$$

and the corresponding particular solutions

$$y_0 = Ae^{it} + A^*e^{-it},$$

$$y_1 = \frac{1}{2}A(1-|A|^2)te^{it} + \frac{i}{8}A^3e^{3it} + \text{c.c.},$$

$$y_2 = \tfrac{1}{8}A(1-|A|^2)(1-3|A|^2)t^2e^{it} - \tfrac{1}{16}iA(2-8|A|^2+7|A|^4)te^{it} - \tfrac{3}{16}iA^3(|A|^2-1)te^{3it} - \tfrac{1}{64}A^3(|A|^2+2)e^{3it} - \tfrac{5}{192}A^5e^{5it} + \text{c.c.},$$

$$y_3 = -\tfrac{1}{48}A(-27|A|^4+15|A|^6-1+13|A|^2)t^3e^{it} + \tfrac{1}{32}iA(21|A|^6 + 18|A|^2 - 2 - 37|A|^4)t^2e^{it} - \tfrac{1}{128}A|A|^2(-70|A|^2 + 32 + 37|A|^4)te^{it} + \tfrac{3}{64}iA^3(-8|A|^2+3+5|A|^4)t^2e^{3it} + \tfrac{1}{128}A^3|A|^2(26|A|^2-23)te^{3it} + \tfrac{1}{512}iA^3(29|A|^4-42|A|^2 + 4)e^{3it} + \tfrac{25}{384}A^5(|A|^2-1)te^{5it} - \tfrac{5}{4608}iA^5(14+3|A|^2)e^{5it} - \tfrac{7}{1152}iA^7e^{7it} + \text{c.c.}. \tag{16}$$

Despite the complexity of the above particular solutions, with a little thought it can be seen that the incoherent secular terms can be collected into secular series expressions so that the naive perturbation expansion can be written in the illuminating form

$$y = \mathcal{A}e^{it} + \epsilon\frac{i}{8}\mathcal{A}^3e^{3it} - \epsilon^2\left(\frac{1}{64}\mathcal{A}^3(|\mathcal{A}|^2+2)e^{3it} + \frac{5}{192}\mathcal{A}^5e^{5it}\right) + \epsilon^3\left(\frac{1}{512}i\mathcal{A}^3(29|\mathcal{A}|^4-42|\mathcal{A}|^2+4)e^{3it} - \frac{5}{4608}i\mathcal{A}^5(14+3|\mathcal{A}|^2)e^{5it} - \frac{7}{1152}i\mathcal{A}^7e^{7it}\right) + \text{c.c.}, \tag{17}$$

where

$$\mathcal{A} = A[1 + \epsilon\tfrac{1}{2}(1-|A|^2)t + \epsilon^2\tfrac{1}{8}(1-|A|^2)(1-3|A|^2)t^2 - \epsilon^2\tfrac{1}{16}i(2-8|A|^2+7|A|^4)t + O(\epsilon^3)], \tag{18}$$

and the secular series y_p is the post factor in the square brackets of the right-hand side of (18). The reader may verify that upon substitution of $\mathcal{A}$ into (17) one recovers all the secular terms in the hierarchy (16). It only remains to find a suitable expression for the slowly varying amplitude $\mathcal{A}$. To this end, we calculate the cumulants of the sequence $\{y_{ip}\}_{i=1}^{\infty}$ and substitute into the algebraic relation (6) for $\mathcal{A}$. Differentiating with respect to time t and applying the near-identity transformation $A = \mathcal{A}y_p^{-1}$ [i.e., the inverse of (18)] leads to the amplitude equation

$$\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{dt} = \epsilon\frac{1}{2}(1-|\mathcal{A}|^2) - \epsilon^2\frac{1}{16}i(7|\mathcal{A}|^4 - 8|\mathcal{A}|^2+2) - \epsilon^3\frac{1}{128}|\mathcal{A}|^2(32-70|\mathcal{A}|^2+37|\mathcal{A}|^4) + O(\epsilon^4).$$

A solution to such an equation can be obtained by standard techniques, first reverting to the time scale $\tau = \epsilon t$ and plane polar coordinates and solving the resulting equations using a regular perturbation method. Finally, applying initial conditions we can obtain the well-known solutions, explicitly derived in the literature [8].

C. Duffing equation

This is the classical analog of the quantum mechanical anharmonic oscillator with cubic nonlinearity. This equation, as is derived in [9], has the form

$$\ddot{y} + y + \epsilon y^3 = 0, \quad t \geqslant 0, \quad y(0) = 1, \quad \dot{y}(0) = 0, \quad \epsilon \to 0+. \tag{19}$$

As this problem was fully solved employing (1) in Ref. [5] we will here illustrate our technique based on (6) and compare with previously obtained results. To this end, expanding the solution in a power series of ϵ leads to a hierarchy of equations and the corresponding particular solutions [relations (72)–(75) of [5]]. These terms can be orderly collected into the secular series expressions while the nonsecular terms lead to the following form of the asymptotic expansion:

$$y = \mathcal{A}e^{it} + \epsilon\tfrac{1}{8}\mathcal{A}^3 e^{3it} + \epsilon^2\left(-\tfrac{21}{64}\mathcal{A}^3|\mathcal{A}|^2 e^{3it} + \tfrac{1}{64}\mathcal{A}^5 e^{5it}\right) + \epsilon^3\left(\tfrac{417}{512}\mathcal{A}^3|\mathcal{A}|^4 e^{3it} - \tfrac{43}{512}\mathcal{A}^5|\mathcal{A}|^2 e^{5it} + \tfrac{1}{512}\mathcal{A}^7 e^{7it}\right),$$

for the amplitude

$$\mathcal{A} = A\left[1 + \epsilon\tfrac{3}{2}i|A|^2 t - \epsilon^2\left(\tfrac{9}{8}t^2 + \tfrac{15}{16}it\right)|A|^4 + \epsilon^3\left(\tfrac{45}{32}t^2 + \tfrac{123}{128}it - \tfrac{9}{16}it^3\right)|A|^6 + O(\epsilon^4)\right], \tag{20}$$

where the secular series y_p is the post factor in the square brackets to the right-hand side of the above expression. For this example we carry out the calculation of the amplitude $\mathcal{A}$ as it follows directly from the algebraic equation (6). Calculating the cumulants of the sequence $\{y_{ip}\}_{i=1}^{\infty}$ from (20) and substituting into the algebraic relation (6) we obtain

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + it\left(\epsilon\tfrac{3}{2}|A|^2 - \epsilon^2\tfrac{15}{16}|A|^4 + \epsilon^3\tfrac{123}{128}|A|^6\right) + O(\epsilon^4). \tag{21}$$

However, on inverting (20) and calculating its modulus, all terms up to the fourth order vanish, i.e., $A = \mathcal{A}[1 + O(\epsilon^4)]$. Thus substituting into the algebraic relation (21) we obtain

$$\ln \mathcal{A} = \ln \mathcal{A}(0) + it\left(\epsilon\tfrac{3}{2}|\mathcal{A}|^2 - \epsilon^2\tfrac{15}{16}|\mathcal{A}|^4 + \epsilon^3\tfrac{123}{128}|\mathcal{A}|^6\right) + O(\epsilon^4). \tag{22}$$

Reverting to plane-polar coordinates $\mathcal{A} = \mathcal{R}e^{i\vartheta}$ provides the following pair of algebraic equations for the polar amplitude and phase:

$$\ln \mathcal{R}(t) = \ln \mathcal{R}(0) + O(\epsilon^4), \tag{23}$$

$$\vartheta(t) = \vartheta(0) + \left(\epsilon\tfrac{3}{2}\mathcal{R}^2 - \epsilon^2\tfrac{15}{16}\mathcal{R}^4 + \epsilon^3\tfrac{123}{128}\mathcal{R}^6\right)t + O(\epsilon^4), \tag{24}$$

which are identical to the result derived by employing the transformation (1) in Ref. [5]. Applying initial conditions we finally obtain well-known solutions explicitly derived in the literature [8].

IV. CONCLUDING REMARKS

In our effort to simplify the renormalization group approach introduced in [2] we derived the amplitude equations without resorting to the process of elimination of secular terms. We only considered the secular series known from solution of the hierarchy of equations. We believe this simplification is close to the message that Ref. [2] wanted to convey and as such it may provide an additional motivation for the use of the renormalization group method.

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